A reductive group with finitely generated cohomology algebras.

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Abstract

Let G be the linear algebraic group SL_3 over a field k of characteristic two. Let A be a finitely generated commutative k-algebra on which G acts rationally by k-algebra automorphisms. We show that the full cohomology ring $H^*(G, A)$ is finitely generated. This extends the finite generation property of the ring of invariants A^G. We discuss where the problem stands for other geometrically reductive group schemes.

1 Introduction

Consider a linear algebraic group scheme G defined over a field k of positive characteristic p. So G is an affine group scheme whose coordinate algebra $k[G]$ is finitely generated as a k-algebra. We say that G has the cohomological finite generation property (CFG) if the following holds. Let A be a finitely generated commutative k-algebra on which G acts rationally by k-algebra automorphisms. (So G acts on $\mathrm{Spec}(A)$.) Then the cohomology ring $H^*(G, A)$ is finitely generated as a k-algebra. Here, as in [8, I.4], we use the cohomology introduced by Hochschild, also known as 'rational cohomology'.

In [18] we have shown that SL_2 over a field of positive characteristic has property (CFG). In this paper we explain how to modify the argument so that it also covers SL_3 in characteristic two. We further explain what is missing to make further progress along the same lines.

Recall that there is no analogous problem in characteristic zero, because rational cohomology vanishes in higher degrees for any linearly reductive group.

2 Geometric reductivity

Let G be a linear algebraic group scheme defined over the field k. We first list a number of properties that G may or may not have. They are more familiar for linear algebraic groups than for linear algebraic group schemes, but we will need them in the greater generality.

If G acts rationally by k-algebra automorphisms on a commutative k-algebra A, we will simply say that G acts on A.

Property (FG) Whenever G acts on a finitely generated k-algebra A, the ring of invariants A^G is a finitely generated k-algebra.

Property (Noeth) Whenever G acts on a finitely generated k-algebra A, and M is a noetherian A-module on which G acts compatibly (so the structure map $A \otimes M \to M$ is a map of G-modules), the module of invariants M^G is noetherian over A^G.

Property (Int) Whenever G acts on a finitely generated k-algebra A, leaving invariant an ideal J, the ring of invariants $(A/J)^G$ is integral over the image of A^G.

Remark 2.1 In property (Int) one may drop the condition that A is finitely generated. The resulting property is equivalent to the original.

Property (GR) (Geometric reductivity) Whenever V is a finite dimensional G-module with one dimensional submodule L on which G acts trivially, there is a homogeneous G-invariant polynomial function f on V with $f(0) = 0$, $f|_L \neq 0$.

Property (GRN) Whenever V is a finite dimensional G-module with one dimensional submodule L on which G acts trivially, the coordinate algebra $k[L]$ is a noetherian $k[V]^G$-module. Here $k[V]$ is of course the coordinate algebra of V.

Property (GRI) Whenever V is a finite dimensional G-module with one dimensional submodule L on which G acts trivially, the coordinate algebra $k[L]$ is integral over the image of $k[V]^G$.

We may now summarize some invariant theory as follows

Theorem 2.2 *The properties (FG), (Noeth), (Int), (GR), (GRN), (GRI) are equivalent.*

Proof That (GR), (GRN), (GRI) are equivalent is pretty obvious, as $k[L]$ is just a polynomial ring in one variable, and the restriction map $k[V] \to k[L]$ is a surjective map of graded rings. It is a theorem of Nagata [10] that (GR) implies (FG). (By [1] the argument also works when G is a group scheme.) That (FG) implies (Noeth) follows by considering the symmetric algebra $S_A(M)$ on M over A, or by considering its quotient ring with underlying set $A \oplus M$. The rest is easier. $\square$

By k-algebra we always mean an associative ring that contains k in its center.

Lemma 2.3 *Let A be a k-algebra and l/k a field extension. Then A is finitely generated as a k-algebra if and only if $A \otimes_k l$ is finitely generated as an l-algebra.*

Proof Field extensions are faithfully flat. $\square$

Encouraged by this lemma we further assume that k is algebraically closed.

Lemma 2.4 *Let G be a finite group scheme over k. Let it act on a commutative k-algebra A. Then A is integral over A^G. In particular, G is geometrically reductive.*

Proof In view of Theorem 2.2 the last statement may be left as an exercise. When G is reduced, hence just a finite group, the lemma is classical: a is a root of $\prod_{g \in G}(x - g(a))$. And if G is not reduced, it has its first Frobenius kernel G_1 as a normal subgroup and $A^G = (A^{G_1})^{G/G_1}$. Now the Lie algebra [8, I,7.7] acts by derivations on A and G_1 acts trivially on the subalgebra A^p generated by p-th powers. By induction on the rank $\dim_k(k[G])$ we may assume A^{G_1} is integral over A^G. $\square$

2.1 Reductive

By Haboush [8, II.10.7] reductive groups are geometrically reductive. By Popov [12] a geometrically reductive linear algebraic group has to be reductive, and by Waterhouse [20] a linear algebraic group scheme G is geometrically reductive exactly when the connected component G^o_{red} of its reduced subgroup G_{red} is geometrically reductive.

3 Cohomological variations

We keep our field k algebraically closed of positive characteristic p. We now introduce cohomological variants of the properties (FG), (Noeth), (Int). We do not know how to make cohomological variants of (GR), (GRI), (GRN).

Property (CFG) Whenever G acts on a finitely generated k-algebra A, the cohomology ring $H^*(G, A)$ is a finitely generated k-algebra.

Property (CNoeth) Whenever G acts on a finitely generated k-algebra A, and M is a noetherian A-module on which G acts compatibly, the cohomology $H^*(G, M)$ is noetherian as a module over $H^*(G, A)$.

Property (CInt) Whenever G acts on a finitely generated k-algebra A, leaving invariant an ideal J, the even part $H^{\mathrm{even}}(G, A/J)$ of the cohomology ring $H^*(G, A/J)$ is integral over the image of $H^{\mathrm{even}}(G, A)$.

Remark 3.1 It is not essential to restrict to the even part, but the advantage of that is that the even part is a commutative ring, so that we do not have to explain the terminology 'integral'. In any case, we will see that it is the even part that truly matters. Note that $H^*(G, A)$ is a finitely k-algebra if and only if $H^{\mathrm{even}}(G, A)$ is a finitely generated k-algebra and $H^*(G, A)$ is a noetherian module over it.

Remark 3.2 In property (CInt) one may drop the condition that A is finitely generated. The resulting property is equivalent to the original.

Lemma 3.3 *Property (CFG) implies properties (CNoeth) and (Cint).*

Proof Apply the same reasoning as in the proof of 2.2. □

Theorem 3.4 (Evens 1961 [3]) *A finite group has property (CFG).*

Note that this is stronger than the theorem of Venkov [19], which refers only to trivial coefficients. Our interest is in nontrivial actions on finitely generated algebras.

After many attempts the result of Evens was generalized by Friedlander and Suslin, but they forgot to make it explicit in the same generality as Evens did.

Theorem 3.5 (Friedlander and Suslin 1997 [5]) *A finite group scheme has property (CFG).*

Proof If G is connected, take $C = A^G$ in [5, Theorem 1.5, 1.5.1]. If G is not connected, one finishes the argument by following [3] as on pages 220–221 of [5]. More specifically, on page 221 of [5] one wants to replace 'the subalgebra of $H^*(G, k)$ generated by the η_i' with 'the C-subalgebra of $H^*(G, C)$ generated by the η_i', where $C = A^G$ again. □

Lemma 3.6 *Let G be a linear algebraic group and H a geometrically reductive subgroup scheme of G. Then G/H is affine.*

Proof If H is reduced, see Richardson [13]. If not, choose r so large that the image of H under the r-th Frobenius homomorphism $F^r : G \to G^{(r)}$ of [8, I.9.4] is reduced. This image is then a geometrically reductive linear algebraic subgroup of the linear algebraic group $G^{(r)}$. The map $G/H \to F^r(G)/F^r(H)$ is finite [2, III,§3,5.5b], hence affine, so G/H is affine because $F^r(G)/F^r(H)$ is affine. □

Now recall from [18] the following suggestive result.

Lemma 3.7 *Let G be a linear algebraic group with property (CFG). Then any geometrically reductive subgroup scheme H of G also has property (CFG).*

Proof The quotient G/H is affine, hence there is no cohomology of quasi-coherent sheaves to worry about. By [8, I 4.6, I 5.13] one simply gets, when H acts on a finitely generated A, that $H^*(H, A) = H^*(G, \mathrm{ind}_H^G A)$. And $\mathrm{ind}_H^G A = (k[G] \otimes A)^H$ is finitely generated. □

For instance, if GL_n as a group scheme over k satisfies (CFG), then so does SL_n. Conversely, if SL_n satisfies (CFG), then so does GL_n, because $H^*(GL_n, A) = H^*(SL_n, A)^{\mathbb{G}_m}$ for any GL_n-algebra A. Further every linear algebraic group scheme is a subgroup scheme of some GL_n—hence of some SL_{n+1}—and it is natural to ask

3.1 Problem

Prove that the linear algebraic group SL_n over k has property (CFG).

We now recall a principle that is a key ingredient in the proofs of theorems 3.4 and 3.5. First a definition.

Definition 3.8 We say that a spectral sequence

$$E_r \Rightarrow \text{Abutment}$$

stops if there is an integer r_0 so that the differential $d_r : E_r \to E_r$ vanishes for $r \geq r_0$.

We now formulate the principle as a slogan.

Lemma 3.9 *A noetherian spectral sequence stops.*

Let us explain what we mean by that. Assume an associative ring R acts on the levels of a spectral sequence

$$E_r \Rightarrow \text{Abutment}$$

in such a manner that each differential $d_r : E_r \to E_r$ is an R-module map and each isomorphism $\ker d_r / \operatorname{im} d_r \cong E_{r+1}$ is R-linear. Say the spectral sequence starts at level two. Assume that E_2 is a noetherian R-module. Then the spectral sequence stops. The proof is easy. One writes E_r as Z_r/B_r where Z_r, B_r are the appropriate submodules of E_2. As the B_r form an ascending sequence, we must have an s so that $B_r = B_s$ for $r \geq s$. Then d_r vanishes for $r \geq s$.

In the examples where the lemma is applied one often has a multiplicative structure on the spectral sequence and the R-module structure of E_2 comes from a graded ring map $R \to E_2$. Say E_2 is a finitely generated k-algebra, which we think of as given. Then to show that E_2 is a noetherian R-module one wants to see that the image of R in E_2 is big enough. Indeed the subring $E_2^{\mathrm{even}} \cap \mathrm{image(R)}$ should be so big that E_2^{even} is integral over it. Thus the paradoxical situation is that in order to prove finite generation results (for an abutment) one needs to exhibit enough images (of elements of R in E_2). That is why we will be looking for universal cohomology classes (to take cup product with).

3.2 The Grosshans family

From now on let G be the linear algebraic group SL_n over k. If G acts on a finitely generated commutative k-algebra A, Grosshans has studied in [6] a flat family over the affine line with general fiber A and special fiber $\operatorname{gr} A$. Here $\operatorname{gr} A$ is of course the associated graded ring with respect to a certain filtration. We write $\mathcal{A}$ for the finitely generated k-algebra whose spectrum is the total space of the family. In characteristic zero the family had been studied earlier by Popov, and in characteristic p a variant of $\operatorname{gr} A$ had been crucial in work of Mathieu [9]. It was the more recent book [7] that alerted us to possible relevance of the family in the present context.

3.3 Good filtrations

We choose a Borel group $B^+ = TU^+$ of upper triangular matrices and the opposite Borel group B^-. The roots of B^+ are positive. If $\lambda \in X(T)$ is dominant, then $\mathrm{ind}_{B^-}^G(\lambda)$ is the 'dual Weyl module' or 'costandard module' $\nabla_G(\lambda)$ with highest weight λ. The formula $\nabla_G(\lambda) = \mathrm{ind}_{B^-}^G(\lambda)$ just means that $\nabla_G(\lambda)$ is obtained from the Borel-Weil construction: $\nabla_G(\lambda)$ equals $H^0(G/B^-, \mathcal{L})$ for a certain line bundle on the flag variety G/B^-. In a good filtration of a G-module the layers are traditionally required to be of the form $\nabla_G(\mu)$. However, to avoid irrelevant contortions when dealing with infinite dimensional G-modules, it is important to allow a layer to be a direct sum of any number of copies of the same $\nabla_G(\mu)$. We always follow that convention (compare [8, II.4.16 Remark 1]). We refer to [17] and [8] for proofs of the main properties of good filtrations. If M is a G-module, and $m \geq -1$ is an integer so that $H^{m+1}(G, \nabla_G(\mu) \otimes M) = 0$ for all dominant μ, then we say

as in [4] that M has *good filtration dimension* at most m. The case $m = 0$ corresponds with M having a good filtration. And for $m \geq 0$ it means that M has a resolution

$$0 \to M \to N_0 \to \cdots \to N_m \to 0$$

in which the N_i have good filtration. We say that M has good filtration dimension precisely m, notation $\dim_\nabla(M) = m$, if m is minimal so that M has good filtration dimension at most m. In that case $H^{i+1}(G, \nabla_G(\mu) \otimes M) = 0$ for all dominant μ and all $i \geq m$. In particular $H^{i+1}(G, M) = 0$ for $i \geq m$. If there is no finite m so that $\dim_\nabla(M) = m$, then we put $\dim_\nabla(M) = \infty$.

Lemma 3.10 *Let $0 \to M' \to M \to M'' \to 0$ be exact.*

1. $\dim_\nabla(M) \leq \max(\dim_\nabla(M'), \dim_\nabla(M''))$,
2. $\dim_\nabla(M') \leq \max(\dim_\nabla(M), \dim_\nabla(M'') + 1)$,
3. $\dim_\nabla(M'') \leq \max(\dim_\nabla(M), \dim_\nabla(M') - 1)$,
4. $\dim_\nabla(M' \otimes M'') \leq \dim_\nabla(M') + \dim_\nabla(M'')$. □

3.4 Filtering A

If M is a G-module, and λ is a dominant weight, then $M_{\leq\lambda}$ denotes the largest G-submodule all whose weights μ satisfy $\mu \leq \lambda$ in the usual partial order [8, II 1.5]. For example, $M_{\leq 0} = M^G$. Similarly $M_{<\lambda}$ denotes the largest G-submodule all whose weights μ satisfy $\mu < \lambda$. As in [17], we form the $X(T)$-graded module

$$\mathrm{gr}_{X(T)} M = \bigoplus_{\lambda \in X(T)} M_{\leq\lambda}/M_{<\lambda}.$$

We convert it to a $\mathbb{Z}$-graded module through an additive height function $\mathrm{ht} : X(T) \to \mathbb{Z}$ defined by $\mathrm{ht} = 2\sum_{\alpha>0} \alpha^\vee$, the sum being over the positive roots. In other words, we put

$$M_{\leq i} = \sum_{\mathrm{ht}(\lambda) \leq i} M_{\leq\lambda}$$

and then $\mathrm{gr}\, M$ is the associated graded module of the filtration $M_{\leq 0} \subseteq M_{\leq 1} \cdots$. We apply this in particular to our finitely generated commutative

k-algebra with G action A. This construction of $\operatorname{gr} A$ goes back to Mathieu [9]. From Grosshans we learn to look at the specialization maps $\mathcal{A} \to \operatorname{gr} A$ and $\mathcal{A} \to A$ ([18, 4.10, 4.11]).

3.5 Cohomology of the associated graded algebra

Under the technical assumption, hopefully unnecessary, that either $n \le 5$ or $p > 2^n$, we have shown in [18, 3.8] that there is a Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(G/G_r, H^j(G_r, \operatorname{gr} A)) \Rightarrow H^{i+j}(G, \operatorname{gr} A)$$

which stops. Here G_r is an appropriately chosen Frobenius kernel in $G = SL_n$. (The choice of r is not constructive. In contrast to the work of Friedlander and Suslin we get only qualitative results.) The fact that the spectral sequence stops was derived from an estimate of good filtration dimensions, not by lemma 3.9.

From the stopping of the spectral sequence one gets

Theorem 3.11 *[18, Thm 1.1] Let SL_n act on the finitely generated k-algebra A. If $n < 6$ or $p > 2^n$, then $H^*(SL_n, \operatorname{gr} A)$ is finitely generated as a k-algebra.*

The filtration $A_{\le 0} \subseteq A_{\le 1} \cdots$ induces a filtration of the Hochschild complex [8, I.4.14] whence a spectral sequence

$$E(A) : E_1^{ij} = H^{i+j}(G, \operatorname{gr}_{-i} A) \Rightarrow H^{i+j}(G, A).$$

It lives in an unusual quadrant, but as long as the E_1 is a finitely generated k-algebra this causes no difficulty with convergence: given m there will be only finitely many nonzero $E_1^{m-i,i}$. (Compare [18, 4.11]. Note that in [18] the E_1 is mistaken for an E_2.)

Thus theorem 3.11 leads to the question whether the spectral sequence $E(A)$ also stops. If it does, then we have no difficulty in transferring the finite generation from E_1 to abutment. (As the differentials are derivations, the p-th power of a homogeneous element of even degree in E_r automatically passes to E_{r+1}. That makes that E_{r+1} is finitely generated when E_r is.) As ring of operators to act on this spectral sequence we may take $R = H^*(G, \mathcal{A})$, with $\mathcal{A}$ the coordinate ring of the Grosshans family ([18, 4.11, 4.12]). The trick is then to show that the natural map $R \to E_1$ makes E_1 into a noetherian module. If SL_n has property (CNoeth) or (CInt), then this is clear because $\operatorname{gr} A$ is a quotient of $\mathcal{A}$. We conclude

Theorem 3.12 *If $n < 6$ or $p > 2^n$, then properties (CFG), (CNoeth), (CInt) are equivalent for SL_n.*

To show that E_1 is a noetherian R-module we wish to have some idea of the size of the image of R in E_1, and also of the size of E_1 itself. Our E_1 is the abutment of the earlier Hochschild-Serre spectral sequence. So we look back at that spectral sequence and try to get it also noetherian over R, even though we already know it stops! Looking at the proof of its stopping we see that actually the Hochschild-Serre spectral sequence is noetherian over $H^{\mathrm{even}}(G, \operatorname{gr} A)$. So it would follow from property (CInt) that the Hochschild-Serre spectral sequence is indeed noetherian over R. But let us look more directly at the image of $R^{\mathrm{even}} = H^{\mathrm{even}}(SL_n, \mathcal{A})$ in $H^0(G/G_r, H^*(G_r, \operatorname{gr} A))$.

3.6 Two module structures

Friedlander and Suslin did not just show that $H^*(G_r, \operatorname{gr} A)$ is a finitely generated k-algebra, they actually provide an explicit finitely generated graded k-algebra $\mathcal{S}^*$ with G action so that a natural graded map $(\mathcal{A})^{G_r} \otimes \mathcal{S}^* \to H^*(G_r, \operatorname{gr} A)$ makes the target into a noetherian module over the source. (We suppress in the notation that $\mathcal{S}^*$ depends on n and r.) Here the subgroup scheme G_r acts trivially on $\mathcal{S}^*$, so that we also have an action of G/G_r on $(\mathcal{A})^{G_r} \otimes \mathcal{S}^*$. Then of course $H^0(G/G_r, H^*(G_r, \operatorname{gr} A))$ is also a noetherian $H^0(G/G_r, (\mathcal{A})^{G_r} \otimes \mathcal{S}^*)$-module. Thus we like to compare the images of R^{even} and $H^0(G, (\mathcal{A})^{G_r} \otimes \mathcal{S}^*) = H^0(G/G_r, (\mathcal{A})^{G_r} \otimes \mathcal{S}^*)$ in $H^0(G/G_r, H^*(G_r, \operatorname{gr} A))$. More specifically, we would like the image of R^{even} to contain the image of $H^0(G, (\mathcal{A})^{G_r} \otimes \mathcal{S}^*)$. That will prove that the E_2 of the Hochschild-Serre spectral sequence is a noetherian R-module (cf. [18, Prop. 3.8]). It suffices to factor the map $H^0(G, (\mathcal{A})^{G_r} \otimes \mathcal{S}^{2m}) \to H^0(G/G_r, H^{2m}(G_r, \operatorname{gr} A))$ through $H^{2m}(G, \mathcal{A})$. This is why one would like to have universal classes in $H^{2m}(G, (\mathcal{S}^{2m})^\#)$, where we use $V^\#$ as a notation for the linear dual of a vector space V. Taking cup product with such a class yields a map from $H^0(G, (\mathcal{A})^{G_r} \otimes \mathcal{S}^{2m})$ to $H^{2m}(G, (\mathcal{A})^{G_r})$ and then one has to show that this map fits in the obvious way in a commutative diagram. This then necessitates putting restrictions on the universal class.

4 Universal cohomology classes

Having explained why we care about having certain universal cohomology classes, let us now get more specific. Further details are given in [18]. Recall that $G = SL_n$ over an algebraically closed field k of positive characteristic p.

Let $W_2(k) = W(k)/p^2W(k)$ be the ring of Witt vectors of length two over k, see [14, II §6]. If V is a module for G (or for GL_n) we write $V^{(r)}$ for the r-th Frobenius twist of V. One has an extension of algebraic groups

$$1 \to \mathfrak{gl}_n^{(1)} \to GL_n(W_2(k)) \to GL_n(k) \to 1,$$

whence a cocycle class $e_1 \in H^2(GL_n, \mathfrak{gl}_n^{(1)})$. We call it the Witt vector class for GL_n. Its m-th cup power defines an element $e_1^{\cup m}$ in $H^{2m}(G, (\mathfrak{gl}_n^{(1)})^{\otimes m})$. Now $(\mathfrak{gl}_n^{(1)})^{\otimes m}$ contains the divided power $\Gamma^m(\mathfrak{gl}_n^{(1)})$ as the submodule of invariants under permutation of the m factors. We wish to lift $e_1^{\cup m}$ to a class in $H^{2m}(G, \Gamma^m(\mathfrak{gl}_n^{(1)}))$, in such a manner that some natural properties are satisfied. Recall that we have for $i \geq 1$, $j \geq 1$ an inclusion $\Delta_{i,j} : \Gamma^{i+j}(\mathfrak{gl}_n^{(1)}) \to \Gamma^i(\mathfrak{gl}_n^{(1)}) \otimes \Gamma^j(\mathfrak{gl}_n^{(1)})$.

Definition 4.1 We call the pair of integers a, b *special* if there is $i \geq 0$ with $a = p^i$ and $1 \leq b \leq (p-1)a$. If a, b is a special pair, then so is ap^s, bp^s for any $s \geq 0$. Any integer larger than one can be written uniquely as $a+b$ with a, b special.

4.1 Lifting Problem

Put $c[1] = e_1$. Show that there are $c[m] \in H^{2m}(G, \Gamma^m(\mathfrak{gl}_n^{(1)}))$ so that for every special pair a, b one has

$$(\Delta_{a,b})_*(c[a+b]) = c[a] \cup c[b]$$

in $H^{2(a+b)}(G, \Gamma^a(\mathfrak{gl}_n^{(1)}) \otimes \Gamma^b(\mathfrak{gl}_n^{(1)}))$.

Remark 4.2 In [18, Thm 4.4] we required $(\Delta_{a,b})_*(c[a+b]) = c[a] \cup c[b]$ for all pairs a, b with $a \geq 1$, $b \geq 1$.

Remark 4.3 Look at the restriction of $c[m]$ to $H^{2m}(G_1, \Gamma^m(\mathfrak{gl}_n^{(1)})) = H^{2m}(G_1, k) \otimes \Gamma^m(\mathfrak{gl}_n^{(1)})$. The ring $\oplus_{m \geq 0} H^{\mathrm{even}}(G_1, k) \otimes \Gamma^m(\mathfrak{gl}_n^{(1)})$ is a divided power algebra over $H^{\mathrm{even}}(G_1, k)$ and the restriction of $c[m]$ to G_1 is the m-th divided power of the restriction of e_1.

If one has solved the lifting problem and $n \leq 5$ or $p > 2^n$, then one can construct the universal classes in $H^{2m}(G, (\mathcal{S}^{2m})^\#)$ alluded to above and thus establish that G has property (CFG). This is what we did in [18] for $G = SL_2$, except that—as mentioned in the remark 4.2—we gave the $c[m]$ some more properties than we require now. So we must have another look at the proof of [18, Lemma 4.7]. In this proof we must now take a, b special. The multiplication $S^a(\mathfrak{gl}_n^{\#(r)}) \otimes S^b(\mathfrak{gl}_n^{\#(r)}) \to S^{a+b}(\mathfrak{gl}_n^{\#(r)})$ is surjective, so that it still follows by induction that the cup product with $c_i[a+b]^{(r-i)}$ takes the desired values. As for the second part of the lemma, the restriction to $H^{2p^{r-1}}(G_1, \mathfrak{gl}_n^{(r)})$ is the same as before by remark 4.3, [18, 4.6]. All in all the proofs in [18] generalize as soon as one has solved the lifting problem and $n \leq 5$ or $p > 2^n$. But we do not have a method to decide the lifting problem in general.

In our solution of the lifting problem for SL_2 in [18] we used that the unipotent radical of a Borel subgroup is abelian. That may be true for SL_2, but it fails for SL_3, so we have to argue differently. Put $\rho^\vee = \sum_{\alpha \text{ simple}} \alpha^\vee$, so that $\rho^\vee(\varpi_i) = 1$ for every fundamental weight ϖ_i. The following proposition provides the missing link to deal with SL_3 in characteristic two. It also gives a new proof for the SL_2 case.

Proposition 4.4 *Let $n = 2$ or let $n = 3$ and $p = 2$. Let $m \geq 0$. Let M be a G-module with $\rho^\vee(\lambda) \leq m$ for every weight λ of M. Then the good filtration dimension $\dim_\nabla(M^{(1)})$ of $M^{(1)}$ is at most m.*

Corollary 4.5 *Let $n = 2$ or let $n = 3$ and $p = 2$. The maps*

$$(\Delta_{a,b})_* : H^{2(a+b)}(G, \Gamma^{a+b}(\mathfrak{gl}_n^{(1)})) \to H^{2(a+b)}(G, \Gamma^a(\mathfrak{gl}_n^{(1)}) \otimes \Gamma^b(\mathfrak{gl}_n^{(1)}))$$

are surjective for $a \geq 1$, $b \geq 1$, so that the $c[m]$ of problem 4.1 exist and property (CFG) holds for G.

Proof The cokernel C of $\Delta_{a,b}$ satisfies $H^{2(a+b)}(G, C) = 0$, because $\dim_\nabla(C) < 2(a+b)$. □

Proof of Proposition This is rather straightforward. One starts with checking that the cokernel of $\nabla(\varpi_i)^{(1)} \to \nabla(p\varpi_i)$ has a good filtration for each fundamental weight ϖ_i. This fails for other values of n, p. Let us use

induction on m. The case $m = 0$ of the proposition is clear. Let $m > 0$ and assume the result for strictly smaller values. We may assume M is finite dimensional and, taking a composition series, we may assume M is a simple module $L(\lambda)$ of high weight λ with $\rho^\vee(\lambda) = m$. Choose ϖ_i so that $\mu = \lambda - \varpi_i$ is dominant. Then $\dim_\nabla(\nabla(\mu)^{(1)} \otimes \nabla(\varpi_i)^{(1)}) \le \rho^\vee(\mu) + 1 \le m$ and the cokernel of the embedding $L(\lambda)^{(1)} \to \nabla(\mu)^{(1)} \otimes \nabla(\varpi_i)^{(1)}$ has good filtration dimension strictly less than m by induction. □

Remark 4.6 There is a reason why proposition 4.4 does not hold for other combinations of n, p. We now sketch this reason. If V is the defining representation of GL_n then as another corollary one gets a surjection $H^4(G, \Gamma^2(V^{(1)})) \otimes \Gamma^2(V^{(1)\#})) \to H^4(G, (\mathfrak{gl}_n^{(1)})^{\otimes 2})$. In particular, let $\tau \in H^4(G, \Gamma^2(V^{(1)})) \otimes \Gamma^2(V^{(1)\#}))$ lift $e_1 \cup e_1$. Then cup product with τ describes a component of the homomorphism $S^*((V \otimes V^\#)^\#) \to H^{\mathrm{even}}(G_1, k)$ that describes the connection [16, Thm 5.2] between cohomology of G_1 and the restricted nullcone. (The restricted nullcone consists of the n by n matrices whose p-th power is zero.) But then the restricted nullcone can only contain matrices of rank at most one: Two by two minors, viewed as elements of $S^2((V \otimes V^\#)^\#)$, are annihilated by $\Gamma^2(V)) \otimes \Gamma^2(V^\#)$, viewed as subspace of $(S^2((V \otimes V^\#)^\#)^\#$. That leaves only $n = 2$ or $n = 3$ and $p = 2$.

Corollary 4.7 *(cf. [18, Cor. 4.13]) Let $n = 2$ or let $n = 3$ and $p = 2$. Let A be a finitely generated algebra with good filtration and let M be a noetherian A-module on which G acts compatibly. Then M has finite good filtration dimension.*

Remark 4.8 We find this surprising. We would have expected some smoothness condition on A or some flatness condition on M to be needed. This makes one wonder if a finitely generated algebra with good filtration is always a quotient of a smooth finitely generated algebra with good filtration. Note that the corollary forbids an algebra to be such a quotient when the algebra does not have finite good filtration dimension. In any case, this may be a good place to look when trying to defeat our expectation that SL_n always satisfies (CFG). On the other hand, in a later preprint (`arXiv:math.RT/0405238`) we show that the corollary extends to the cases where $n < 6$ or $p > 2^n$.

Proof of Corollary We repeat the proof of [18, Cor. 4.13]. Let $k[G/U]$ denote the multicone $\mathrm{ind}_U^G k$. Recall that, for finite d, the good filtration

dimension of M is at most d if and only if $H^{d+1}(G, M \otimes k[G/U])$ vanishes. Now note that $M \otimes k[G/U]$ is a noetherian $A \otimes k[G/U]$-module and note that $H^*(G, A \otimes k[G/U])$ is concentrated in degree zero. □

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